The $B \to \pi \nu$ and $B_s \to K \nu$ form factors from 2+1 flavors of domain-wall fermions and relativistic $b$-quarks

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Motivation

A precise determination of $|V_{ub}|$ allows a strong test of the standard model.

The constraint on the apex $(\bar{\rho}, \bar{\eta})$ of the CKM triangle from $|V_{ub}|$ will strengthen tests of the Standard-Model CKM framework.

$$\frac{|V_{ub}|}{|V_{cb}|} = \frac{\lambda}{1 - \lambda^2/2} \sqrt{\bar{\rho}^2 + \bar{\eta}^2}$$

- $\lambda = |V_{ud}|$ known to $\sim 1\%$
- $|V_{cb}|$ known to $\sim 2\%$

Dominant error (yellow ring) comes from the uncertainty of $|V_{ub}|$ ($\sim 6\%$).

There has been a long standing puzzle in the determination of $|V_{ub}|$.

$\sim 3\sigma$ discrepancy between exclusive ($B \rightarrow \pi l\nu$) and inclusive ($B \rightarrow X_u l\nu$) determination.
Exclusive determination of $|V_{ub}|$

$f_+(q^2)$ is crucial for the determination of the CKM matrix element $|V_{ub}|$.

\[
q^2 = m_{B(s)}^2 + m_P^2 - 2m_{B(s)}E_P
\]

The exclusive $B \to \pi l \nu$ semileptonic decay allows the determination of $|V_{ub}|$ via:

\[
\frac{d\Gamma}{dq^2} = \frac{G_F^2}{192\pi^3 m_{B(s)}^3} \left[ (m_{B(s)}^2 + m_P^2 - q^2)^2 - 4m_{B(s)}^2 m_P^2 \right]^{3/2} \times |f_+(q^2)|^2 \times |V_{ub}|
\]

**Goal**

**Experiment**

**Known factor**

**Hadronic part**

**CKM matrix**

- Experiment can only measure the CKM matrix element times hadronic form factor.
- The hadronic form factor must be computed nonperturbatively via lattice QCD.
Form-factor definitions

- Non-perturbative form factors $f_+(q^2)$ and $f_0(q^2)$ parametrize the hadronic matrix element of the $b \to u$ quark flavor-changing vector current $V_\mu$.

$$\langle P|V_\mu|B_s\rangle = f_+(q^2) \left( p^\mu_{B(s)} + p^\mu_P - \frac{m^2_{B(s)} - p^2_P}{q^2} q^\mu \right) + f_0(q^2) \frac{m^2_{B(s)} - p^2_P}{q^2} q^\mu$$

- On the lattice, we calculate the form factors $f_{||}$ and $f_\perp$.
  - Proportional to vector current matrix elements in the $B(s)$ meson rest frame:

$$f_{||}(E_P) = \frac{\langle P|V_0|B_s\rangle}{\sqrt{2m_{B(s)}}}$$
$$f_\perp(E_P)p_i = \frac{\langle P|V_i|B_s\rangle}{\sqrt{2m_{B(s)}}}$$

- Easy to relate to the desired form factor $f_+(q^2)$ and $f_0(q^2)$.

$$f_0(q^2) = \frac{\sqrt{2m_{B(s)}}}{m^2_{B(s)} - m^2_P} \left[ (m_{B(s)} - E_P)f_{||}(E_P) + (E^2_P - m^2_P)f_\perp(E_P) \right]$$
$$f_+(q^2) = \frac{1}{\sqrt{2m_{B(s)}}} \left[ f_{||}(E_P) + (m_{B(s)} - E_P)f_\perp(E_P) \right]$$
Lattice actions and parameters

- We use the 2+1 flavor dynamical domain-wall fermion gauge field configurations generated by the RBC/UKQCD Collaborations.
  

- For the $b$-quark we use the relativistic heavy quark (RHQ) action developed by Li, Lin, and Christ in Refs. N. H. Christ, M. Li, and H.-W. Lin, Phys.Rev. D76, 074505 (2007)

- We use the nonperturbative determinations of the parameters of the RHQ action obtained in Y.Aoki et. al Phys. Rev. D 86, 116003 (2012).

- Provides important cross-check of existing $N_f = 2+1$ calculations using the MILC staggered ensembles.

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<th>L×T</th>
<th>$a$ [fm]</th>
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<th>$m_s$</th>
<th>$m_{\pi}$ [MeV]</th>
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Calculation of lattice form factors

Domain wall fermion action

\[ p \neq 0 \]

\( p_B = 0 \)

Gaussian-smeared sequential source

Relativistic heavy quark action

The 2+1 flavor dynamical domain-wall fermion gauge field configurations

- Extract the lattice form factor from the ratio of the 3pt function to 2pt functions:

  \[
  R_{3,\mu}^{B(s)\rightarrow P} (t, T) = \frac{C_{3,\mu}^{B(s)\rightarrow P} (t, T)}{\sqrt{C_2^P (t) C_2^{B(s)} (T - t)}} \sqrt{\frac{2E_P}{e^{-E_P t} e^{-m_{B(s)} (T-t)}}} 
  \]

  \[
  f_{\text{lat}}^\| = \lim_{t,T \rightarrow \infty} R_0^{B(s)\rightarrow P} (t, T) 
  \]

  \[
  f_{\text{lat}}^\perp = \lim_{t,T \rightarrow \infty} \frac{1}{p_P} R_{i}^{B(s)\rightarrow P} (t, T) 
  \]

Three-point correlator fits

- We use the lattice data up to $(1,1,1)$ for $B \to \pi$ and $(2,0,0)$ for $B_s \to K$.
- After a careful study, we fix source-sink separations $T = t_B - t_\pi$.
- We fit the ratio to a plateau in the region $0 \ll t \ll T$. 
Renormalization of lattice form factors

• The continuum form factors are given by

\[
\begin{align*}
  f_{\parallel}(E_P) &= Z_{V_0}^{bl} \lim_{t,T \to \infty} R_{0}^{B(s)\to P}(E_P, t, T) \\
  f_{\perp}(E_P) &= Z_{V_i}^{bl} \lim_{t,T \to \infty} \frac{1}{p_P^i} R_{i}^{B(s)\to P}(E_P, t, T)
\end{align*}
\]

• We calculate the heavy-light current renormalization factor $Z_{V}^{bl}$ using the mostly nonperturbative method.

  \[Z_{V_{\mu}}^{bl} = \left( \rho_{V_{\mu}}^{bl} \right) \sqrt{Z_{V}^{bb} Z_{V}^{ll}}\]

  compute nonperturbatively

  compute with 1-loop mean-field improved lattice perturbation theory

  \[\approx 1\]


  C. Lehner arXiv:1211.4013

  • $\rho$-factor calculated in PhySyHCAI (framework for automated lattice perturbation theory).


  • $Z_{V}^{ll}$ obtained by the RBC/UKQCD collaborations by exploiting the fact $Z_A = Z_V$ for domain-wall fermions.

  N. H. Christ et al. (RBC/UKQCD Collaboration), arXiv:1404.4670

  • $Z_{V}^{bb}$ obtained from the matrix element of the $b \to b$ vector current between two $Bs$ mesons.
Chiral-continuum extrapolations of $f_{\parallel}$ and $f_{\perp}$

- Correlated simultaneous chiral-continuum fit ($m_{\pi} \to m_{\pi}^{\text{phys}}, a \to 0$)
to $f_{\perp}$ and $f_{\parallel}$ data using Hard-pion NLO SU(2) $\chi$PT.
  - Strange quark integrated out
  - Applies to regime where $E_P >> m_{\pi}$


\[ f_{\parallel}(m_{\pi}, E_P, a^2) = c_{\parallel}^{(1)} \left( 1 + (\delta f_{\parallel})^{\text{Hard-pion}} + c_{\parallel}^{(2)} \frac{m_{\pi}^2}{\Lambda^2} + c_{\parallel}^{(3)} \frac{E_P}{\Lambda} + c_{\parallel}^{(4)} \frac{E_P^2}{\Lambda^2} + c_{\parallel}^{(5)} \frac{a^2}{\Lambda^2 a_{32}^4} \right) \]

\[ f_{\perp}(m_{\pi}, E_P, a^2) = \frac{1}{E + m_{B}^* - m_B} c_{\perp}^{(1)} \left( 1 + (\delta f_{\perp})^{\text{Hard-pion}} + c_{\perp}^{(2)} \frac{m_{\pi}^2}{\Lambda^2} + c_{\perp}^{(3)} \frac{E_P}{\Lambda} + c_{\perp}^{(4)} \frac{E_P^2}{\Lambda^2} + c_{\perp}^{(5)} \frac{a^2}{\Lambda^2 a_{32}^4} \right). \]

The function $\delta f$ indicate non-analytic “log” functions of the pion mass.

- The hard-pion SU(2) logarithms are given by simply taking the limit $m_{\pi}/E_P \to 0$.

\[ (4\pi f_{\pi})^2 (\delta f_{\parallel,\perp})^{B \to \pi}^{\text{Hard-pion}} = -\frac{3}{4} \left( 3g^2 + 1 \right) m_{\pi}^2 \log \left( \frac{m_{\pi}^2}{\Lambda^2} \right) \]

\[ (4\pi f_{\pi})^2 (\delta f_{\perp,\parallel})^{B \to K}^{\text{Hard-pion}} = -\frac{3}{4} m_{\pi}^2 \log \left( \frac{m_{\pi}^2}{\Lambda^2} \right) \]
Chiral-continuum extrapolations of $f_{\parallel}$ and $f_{\perp}$

Black curves show chiral-continuum extrapolated $f_{\parallel}$ and $f_{\perp}$ with statistical errors.
Preliminary error budgets

• Show error budgets for three $q^2$ points within the range of simulated lattice momenta.
• Dominant uncertainties from statistics and chiral extrapolation.
• Estimate error from chiral extrapolation from difference between SU(2) $\chi$PT and analytic fits.

Preliminary

$B \rightarrow \pi l \nu$
Using the output of the chiral-continuum fit, we generate 3 synthetic data points for \( f_+ \) and \( f_0 \) (black) evenly spaced in the range of simulated \( z \) values to use in the extrapolation to \( q^2=0 \).
**z-expansion of $f_+$ and $f_0$**


We employ the model-independent $z$-expansion fit to extrapolate lattice results to full kinematic range.

- Consider mapping the variable $q^2$ onto a new variable $z$.
  - **semileptonic region**
    
    $0 < q^2 < t_-$  $\rightarrow$  $-0.34 < z < 0.22$  
    (when we choose $t_0 = 0.65t_+$)

- The form factor $f(q^2)$ is analytic in the semileptonic region except at $B^*$ pole.
  $\rightarrow$ $f(q^2)$ can be expressed as convergent power series.

\[
f(q^2) = \frac{1}{P(q^2)\phi(q^2, t_0)} \sum_{k=0}^{\infty} a^{(k)}(t_0) z(q^2, t_0)^k
\]

- The sum of the series coefficients is bounded by unitarity.

\[
\sum_{k=0}^{N} a^{(k)}^2 \leq 1
\]

- Therefore this bound combined with the small $|z|$ ensures that only a small number of terms is needed to accurately describe the shape of the form factor.
**z-expansion of $f_+$ and $f_0$**

**$P\psi f_{+/0}$ vs $z$**

<table>
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<tr>
<th># of par.</th>
<th>$a_+^{(0)}$</th>
<th>$a_+^{(1)}$</th>
<th>$a_+^{(2)}$</th>
<th>$a_+^{(3)}$</th>
<th>$\chi^2$/d.o.f</th>
<th>$p$-value</th>
<th>$a_0^{(0)}$</th>
<th>$a_0^{(1)}$</th>
<th>$a_0^{(2)}$</th>
<th>$a_0^{(3)}$</th>
<th>$\chi^2$/d.o.f</th>
<th>$p$-value</th>
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<td>2</td>
<td>0.0231(28)</td>
<td>-0.97(50)</td>
<td>0.92(50)</td>
<td>0.37(50)</td>
<td>0.40</td>
<td>53%</td>
<td>0.077(12)</td>
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<td>0.071(13)</td>
<td>-4.9(1.9)</td>
<td>0.91</td>
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<tr>
<td>3</td>
<td>0.0223(31)</td>
<td>-2.5(2.5)</td>
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<td>—</td>
<td>—%</td>
<td>0.071(13)</td>
<td>-4.9(1.9)</td>
<td>0.071(13)</td>
<td>-4.9(1.9)</td>
<td>—</td>
<td>—%</td>
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</table>

Our data determines normalization and slope, but only loosely constrains curvature.

**$f_{+/0}$ vs $q^2$**

Our data shows that $f_+(q^2 = 0) = f_0(q^2 = 0)$. This work was compared to other calculations from FNAL/MILC and HPQCD collaborations.

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*Preliminary*
Determination of $|V_{ub}|$

Now add experimental data to $z$-fit to obtain $|V_{ub}|$.

- $q^2$ dependence of lattice form factor agrees well with experiment.
- Experimental measurements determine both slope and curvature well.
- Error on normalization (and hence $|V_{ub}|$) saturates with 3-parameter $z$-fit.

| # of par. | $a_+^{(1)}/a_+^{(0)}$ | $a_+^{(2)}/a_+^{(0)}$ | $a_+^{(3)}/a_+^{(0)}$ | $|V_{ub}| \times 10^3$ | $\chi^2$/d.o.f. | $p$-value |
|-----------|----------------|----------------|----------------|----------------|----------------|------------|
| 2+1       | -1.76(22)      |                 |               | 4.20(37)       | 1.42           | 6%         |
| 3+1       | -1.22(19)      | -3.6(1.2)       |               | 3.54(36)       | 1.03           | 42%        |
| 4+1       | -1.32(30)      | -4.0(1.5)       | 4(8)          | 3.53(36)       | 1.06           | 38%        |

Conclusions and future prospects

• We have calculated the $B \to \pi$ and $B_s \to K$ form factors using 2+1 flavor dynamical domain-wall fermion gauge field configurations with relativistic heavy quark action.

• Provide important independent check on existing calculations using staggered light quarks.

• Will present final results for $B \to \pi$ and $B_s \to K$ lattice form factors as coefficients of the $z$-expansion and their correlations.

• $|V_{ub}|$ is determined by combined $z$-fit with experimental data from Babar and Belle to about 10% precision.

Still to do:

• Implement unitarity and heavy-quark constraints on sum of coefficients.

• Compare with result using BCL parameterization.
Backup slides
Dispersion relation and amplitude $Z_\pi$

- The pion energies satisfy the continuum dispersion relation: $E_\pi^2 = |\vec{p}_\pi|^2 + m_\pi^2$

- The pion amplitude $Z_\pi = |\langle 0 | O_\pi | \pi \rangle|$ is independent of momentum
\( O(a) \) improved vector current operator

The heavy-light current operator at tree level is

\[
V_{\mu,0}(x) = \bar{q}(x) \mathcal{O}_{\mu,0} Q(x), \quad \mathcal{O}_{\mu,0} = \gamma_\mu
\]

Four single derivative operators are needed for \( O(a) \) improvement.

\[
\begin{align*}
\mathcal{O}_{1,\mu} &= 2 \overrightarrow{D}_\mu \\
\mathcal{O}_{2,\mu} &= 2 \overleftarrow{D}_\mu \\
\mathcal{O}_{3,\mu} &= 2 \gamma_\mu \gamma_i \overrightarrow{D}_i \\
\mathcal{O}_{4,\mu} &= 2 \gamma_\mu \gamma_i \overleftarrow{D}_i
\end{align*}
\]

The \( O(a) \) improved vector current operator is given by

- temporal (\( \mu = 0 \)): \( \mathcal{O}_{0}^{\text{imp}} = \mathcal{O}_{0,0} + c_3^V \mathcal{O}_{0,3} + c_4^V \mathcal{O}_{0,4} \)
- spatial (\( \mu = i \)): \( \mathcal{O}_i^{\text{imp}} = \mathcal{O}_{i,0} + c_1^V \mathcal{O}_{i,1} + c_2^V \mathcal{O}_{i,2} + c_3^V \mathcal{O}_{i,3} + c_4^V \mathcal{O}_{i,4} \)

Coefficients are determined by 1-loop lattice perturbation theory.
Relativistic heavy quark action for b-quarks

Heavy quark mass introduces discretization errors of $O((ma)^n)$.
- At bottom quark mass, it becomes severe: $m_b \sim 4$ GeV and $1/a \sim 2$ GeV, then $m_b a > O(1)$.

![Relativistic heavy quark action (RHQ action)](attachment:image.png)

- The Fermilab group showed that you can remove all errors of $O((ma)^n)$ by appropriately tuning the parameters of the anisotropic clover action
- Errors are of $O(a^2p^2)$.
- Li, Lin, and Christ showed that the parameters $\{m_0, \zeta, c_P\}$ can be tuned nonperturbatively.
- We use the results for the parameters of the RHQ action obtained for b-quarks in Y. Aoki et. al Phys. Rev. D 86, 116003 (2012)
Renormalization factor $Z_{V}^{bb}$

At tree level, the expression of $Z_{V}^{bb}$ is given by

$$Z_{V}^{bb} = u_0 \exp(M_1), \quad M_1 = \log[1 + \tilde{m}_0], \quad \tilde{m}_0 = \frac{m_0}{u_0} - (1 + 3\zeta)(1 - \frac{1}{u_0})$$

Here $m_0 = 7.80, \ zeta = 3.20, \ u_0 = 0.8757$.

**NP** : $Z_{V}^{bb} = 10.037(34)$

**Tree level** : $Z_{V}^{bb} = 9.993$